

Dually Vertex-Oblique Graphs

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Abstract

A vertex with neighbours of degrees $d_1 \geq \dots \geq d_r$ has *vertex type* (d_1, \dots, d_r) . A graph is *vertex-oblique* if each vertex has a distinct vertex-type. While no graph can have distinct degrees, Schreyer, Walther and Mel'nikov [Vertex oblique graphs, same proceedings] have constructed infinite classes of *super vertex-oblique* graphs, where the degree-types of G are distinct even from the degree types of \overline{G} .

G is vertex oblique iff \overline{G} is; but G and \overline{G} cannot be isomorphic, since self-complementary graphs always have non-trivial automorphisms. However, we show by construction that there are *dually vertex-oblique graphs* of order n , where the vertex-type sequence of G is the same as that of \overline{G} ; they exist iff $n \equiv 0$ or $1 \pmod{4}$, $n \geq 8$, and for $n \geq 12$ we can require them to be split graphs.

We also show that a dually vertex-oblique graph and its complement are never the unique pair of graphs that have a particular vertex-type sequence; but there are infinitely many super vertex-oblique graphs whose vertex-type sequence is unique.

1 Introduction and basic results

Let G be a simple graph on n vertices. A vertex v of degree r , with neighbours of degrees¹ $x_1 \geq \dots \geq x_r$, has *vertex type* $t(v) := (x_1, \dots, x_r)$. G is *vertex-*

¹It is conventional in the literature on degree sequences to list degrees in non-increasing order. We follow this convention here, even though we do not prefer it, because we will

oblique if each vertex has a distinct vertex-type.

The degree of v in \overline{G} (the complement of G) is $\overline{r} := n - 1 - r$. If the degrees of vertices in G are $x_1 \geq \dots \geq x_r, r, y_1 \leq \dots \leq y_{\overline{r}}$, then v is non-adjacent to vertices of degrees $y_1 \leq \dots \leq y_{\overline{r}}$, so its vertex-type in \overline{G} is $\overline{t}(v) = (\overline{y}_1, \dots, \overline{y}_{\overline{r}})$. Thus G is vertex oblique if and only if \overline{G} is.

While no graph can have distinct degrees, Schreyer et al. [10] have constructed infinite classes of vertex-oblique graphs. In fact, their examples are *super vertex-oblique*, with the degree-types of G being distinct even from degree types of \overline{G} .

It is natural to ask whether there are any self-complementary vertex-oblique graphs, but this is impossible because a self-complementary graph always has non-trivial automorphisms [7, 9], obtained by applying twice an (y) isomorphism that maps the graph to its complement. However, in this article we construct infinitely many *dually vertex-oblique graphs*, where the set of vertex-types of G is the same as that of \overline{G} .

Many simple results for self-complementary graphs still hold under the weaker assumption that G and \overline{G} have the same set of vertex-types. In particular, dually vertex-oblique graphs of order n can only exist if $n \equiv 0$ or $1 \pmod{4}$. The main result of this paper is that they exist for all feasible n at least 8:

1. Theorem. *Dually vertex-oblique graphs of order $n > 1$ exist iff n is congruent to 0 or 1 $\pmod{4}$, and $n \geq 8$. \square*

We will make use of the following elementary lemma that is inspired by similar results on self-complementary graphs.

2. Lemma. *Let G be a graph with the same degree sequence as \overline{G} , say $d_1 \geq \dots \geq d_n$. Then:*

A. $d_i + d_{n-i+1} = n - 1$, for $i = 1, \dots, n$.

B. $n \equiv 0$ or $1 \pmod{4}$.

If, moreover, G has the same set of vertex-types as \overline{G} , and there are r_d vertices of degree d , and $s_{x,y}$ edges joining vertices of degrees x and y , then:

discuss degree sequences in Section 5; it is of little importance anyway.

- C. $s_{y,y} + s_{\bar{y},\bar{y}} = \frac{1}{2} \binom{r_y}{2}$, and if $x \neq y$, $s_{x,y} + s_{\bar{x},\bar{y}} = r_x r_y$; in particular, $s_{d,\bar{d}} = \frac{1}{2} r_d^2$ for all d , except if $d = \bar{d} = (n-1)/2$ in which case $s_{d,\bar{d}} = \frac{1}{2} \binom{r_d}{2}$.
- D. r_d is even for all d , except for $r_{(n-1)/2} \equiv 1 \pmod{4}$.

Furthermore, if G is dually vertex-oblique, then:

- E. $r_d < 2d$, $d \neq (n-1)/2$; in particular, there are no isolated or end-vertices.
- F. There must be at least three different degrees in G .

Proof: A. If r_d vertices have degree d in G , then r_d vertices have degree d in \bar{G} and, thus, degree $n-1-d$ in G . So the degree sequence is symmetric about $\frac{1}{2}(n-1)$.

B. The number of edges of G and \bar{G} is the same: $\frac{1}{2} \binom{n}{2} = \frac{1}{4} n(n-1)$, which must be an integer.

C. Since $s_{x,y}$ is determined by the vertex-types of G , it must remain the same in \bar{G} , that is, $s_{x,y} = \bar{s}_{x,y}$; similarly, $s_{\bar{x},\bar{y}} = \bar{s}_{\bar{x},\bar{y}}$. Now a vertex of degree x (or y) in G has degree \bar{x} (or \bar{y}) in \bar{G} . So if there are p unordered pairs of vertices $\{v, w\}$ with $d(v) = x$ and $d(w) = y$, then $p - s_{x,y} = \bar{s}_{\bar{x},\bar{y}} = s_{\bar{x},\bar{y}}$. Since $p = \binom{r_x}{2}$ when $x = y$, and $p = r_x r_y$ otherwise, the result follows.

D. By C, $\frac{1}{2} r_d^2$ must be an integer, so r_d is even for $d \neq \bar{d}$. When there are vertices of degree $(n-1)/2$, by B we must have $n = 4k+1$ for some k . By the first part, there are $2r$ vertices with degree $d < (n-1)/2$, and therefore $2r$ vertices with degree $d > (n-1)/2$, leaving $(4k+1) - 2(2r) = 4(k-r) + 1$ vertices of degree $(n-1)/2$.

E. If G is vertex oblique, the vertices of degree d cannot all be adjacent to vertices of degree \bar{d} only. So $\frac{1}{2} r_d^2 = s_{d,\bar{d}} < d r_d$.

F. Clearly G cannot be regular, so there must be at least two different degrees. If n is odd, the number of different degrees must be odd, by A. Suppose n is even and there are exactly two degrees, say $d < (n-1)/2$ and $\bar{d} > (n-1)/2$. Since G is vertex-oblique, each vertex v of degree d must be adjacent to a distinct number n_v of vertices of degree \bar{d} . Note that $0 \leq n_v \leq d$. If $n_v = 0$ for some v , then in the complement v would be a vertex of degree \bar{d} that is adjacent to all vertices of degree d , so \bar{G} would have no vertex w of degree d with $n_w = 0$. Similar reasoning excludes the case $n_v = d$, so we have $0 < n_v < d$ for every v . This means that there are less than $d < n/2 = r_d$ possible values of n_v , a contradiction. \square

By Lemma 2.B, the smallest possible orders $n > 1$ for a dually vertex-oblique graph are 4 and 5; but by part E these could not have any vertices of degree $d < (n-1)/2$, so no such graphs exist when $n < 8$. We now construct graphs for every $n \equiv 0$ or $1 \pmod{4}$, $n \geq 8$.

2 Construction on $4k$ vertices

A dually vertex-oblique graph on 8 vertices is shown in Figure 1, with the degree and vertex-type displayed next to each vertex. To verify this, one has to check that for every vertex with vertex-type (x_1, \dots, x_r) , there is another vertex that is non-adjacent to vertices of degrees $(7 - x_1, \dots, 7 - x_r)$.

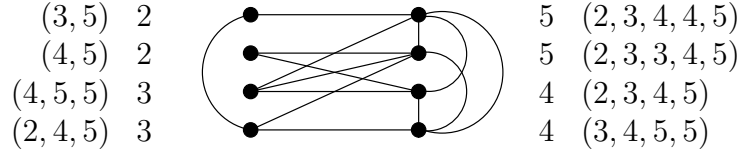


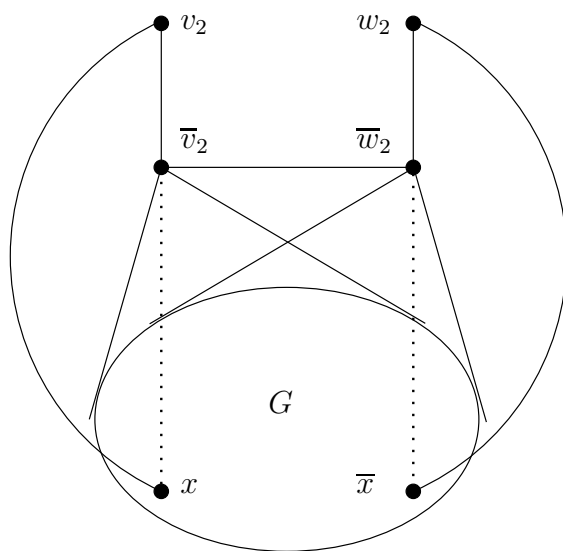
Figure 1: A dually vertex-oblique graph on 8 vertices.

Given a dually vertex-oblique graph G on $n = 4k$ vertices, we now show how to construct G' on $n + 4 = 4k'$ vertices, where $k' := k + 1$ (see Figure 2). We add vertices v_2, w_2 , that will have degree 2, and \bar{v}_2, \bar{w}_2 that will have degree $2k' - 2 = 2k + 2$. Moreover, the new vertices induce a P_4 , in a manner reminiscent of Akiyama and Harary's [1] method of producing larger self-complementary graphs.

We pick an arbitrary vertex $x \in V(G)$, and let \bar{x} be the (unique) vertex such that $\bar{t}(\bar{x}) = t(x)$. Note that if x has degree d , then \bar{x} has degree $4k - 1 - d \neq d$. We make v_2 adjacent to \bar{v}_2 and x , w_2 adjacent to \bar{w}_2 and \bar{x} . Meanwhile, \bar{v}_2 is adjacent to v_2, \bar{w}_2 and $V(G) \setminus x$; and \bar{w}_2 is adjacent to w_2, \bar{v}_2 and $V(G) \setminus \bar{x}$.

A vertex of degree d now has degree $d' := d + 2$; this also means that a vertex of the complementary degree $\bar{d} := 4k - 1 - d$ now still has complementary degree $\bar{d}' = 4k' - 1 - d = \bar{d} + 2$. The degrees in $V(G)$ now range between at least 4 and at most $4k$ (by Lemma 2.E), and thus the degrees (and vertex-types) in $V(G)$ are distinct from those of the new vertices.

If $u \notin \{x, \bar{x}\}$ had vertex-type $t(u) = (d_1, \dots, d_r)$, in G' it has type $t'(u) = (d'_1, \dots, d'_r, 4k' - 2, 4k' - 2)$. The unique vertex \bar{u} such that $\bar{t}(\bar{u}) = t(u)$ was



(dotted lines indicate non-adjacency)

Figure 2: Larger dually vertex-oblique graphs from smaller ones.

non-adjacent in G to vertices of degrees $\overline{d_1}, \dots, \overline{d_r}$; so in G' it is non-adjacent to vertices of degrees $\overline{d'_1}, \dots, \overline{d'_r}$, as well as two vertices of degree $2 = \overline{4k' - 2}$.

Meanwhile $t(x) = \overline{t}(\overline{x}) = (f_1, \dots, f_d)$ becomes $t'(x) = \overline{t'}(\overline{x}) = (2, f_1 + 2, \dots, f_d + 2, 4k' - 2)$. Distinct vertex-types in G therefore result in distinct vertex-types in G' , and complementary vertex-types result in complementary vertex-types. Moreover, $t(v_2) = \overline{t}(\overline{v_2}) = (2, d)$, and $t(w_2) = \overline{t}(\overline{w_2}) = (2, \overline{d})$, so the types of the new vertices are also distinct and complementary, and thus G is dually vertex-oblique.

A graph G is *split* if its vertex set partitions into $L \cup R$ (the “left” and “right” vertices), where $G[L]$ is edgeless and $G[R]$ is complete. Our constructions are close to being split graphs, with the vertex-set partitioning into vertices of degree less than $2k$ and vertices of degree at least $2k$. With a little more effort we can construct examples (for $n \geq 12$) that are actually split graphs.

We will construct an appropriate bipartite graph with partition $L \cup R$ and show that, if we add edges to make R induce a clique, the resulting graph is dually vertex-oblique. If B is bipartite, with partition $L \cup R$, its *bipartite complement* is the graph \tilde{B} with $V(\tilde{B}) := V(B)$, $E(\tilde{B}) := \{uv \mid u \in L, v \in R, uv \notin E(B)\}$. The vertex-type of v in \tilde{B} is $\tilde{t}(v)$. A *dually semi-vertex-oblique* graph is a bipartite graph B with $L = \{\ell_1, \dots, \ell_{2k}\}$, $R = \{r_1, \dots, r_{2k}\}$, such that:

- (i) $\{t(\ell_1), \dots, t(\ell_{2k})\}$ contains no repetitions
- (ii) $\{t(\ell_1), \dots, t(\ell_{2k})\} = \{t(r_1), \dots, t(r_{2k})\}$
- (iii) $\{\tilde{t}(\ell_1), \dots, \tilde{t}(\ell_{2k})\} = \{t(\ell_1), \dots, t(\ell_{2k})\}$, and (thus)
 $\{\tilde{t}(r_1), \dots, \tilde{t}(r_{2k})\} = \{t(r_1), \dots, t(r_{2k})\}$.

If ℓ_1 , say, had degree $2k$, then in \tilde{B} it would have degree 0, so by conditions (ii) and (iii) there must be a vertex of degree 0 in R , a contradiction. So the minimum degree in L is at least 1, and by (iii) the maximum degree is at most $2k - 1$, and similarly for R .

We now add edges to make R induce a clique, giving us a split graph G . The degree of any vertex r_j jumps up by $2k - 1$, so its degree becomes at least $2k$; thus the degrees (and vertex-types) of vertices in R become distinct from those in L . If $t(r_i)$ differed from $t(r_j)$ in the number of entries equal to d , where $d < 2k$, then in G both types get $2k - 1$ new entries that are all

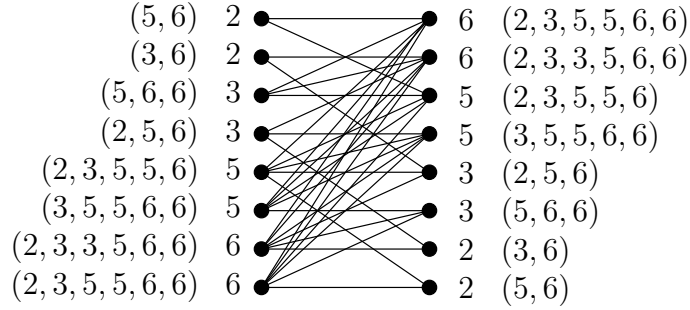
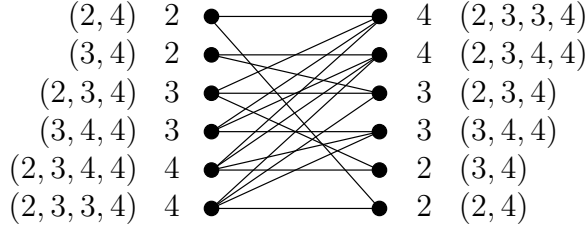


Figure 3: Dually semi-vertex-oblique graphs on 12 and 16 vertices.

at least $2k$, but they still differ in the number of entries equal to d . If $t(\ell_i)$ differed from $t(\ell_j)$ in the number of entries equal to d , in G they will differ in the number of entries equal to $d + 2k - 1$. Thus G is vertex-oblique, and from (iii) we can see that \overline{G} has the same vertex-types as G .

Dually semi-vertex-oblique graphs on 12 and 16 vertices are shown in Figure 3, with their degrees and vertex-types. To verify condition (iii), one has to check that for every vertex on the left with vertex-type (x_1, \dots, x_r) , there is another vertex that is non-adjacent to vertices (on the right) of degrees $(2k - x_1, \dots, 2k - x_r)$.

Given a dually semi-vertex-oblique graph B on $n = 4k$ vertices, we now show how to construct B' on $n + 8 = 4k'$ vertices, where $k' := k + 2$. We add vertices $L_2, L'_2, L_{\tilde{2}}, L'_{\tilde{2}}$ on the left, and $R_2, R'_2, R_{\tilde{2}}, R'_{\tilde{2}}$ on the right. The vertices with subscript 2 will have degree 2, those with subscript $\tilde{2}$ will have degree $2k' - 2 = 2k + 2$. See Figure 4 for a sketch of the new vertices and their adjacencies to each other and to the vertices $\ell, \tilde{\ell}, r, \tilde{r}$ (described below); the vertices with subscript $\tilde{2}$ are also adjacent to all other vertices on the opposite side.

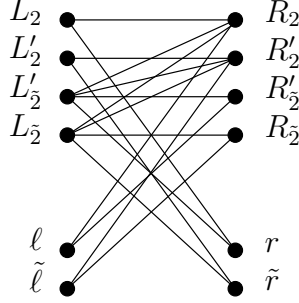


Figure 4: Adding vertices to make larger semi-vertex-oblique graphs.

By (iii) we can find two vertices r, \tilde{r} such that $\tilde{t}(\tilde{r}) = t(r)$; in particular, if r has degree d , then \tilde{r} has degree $2k - d$. We make L_2 adjacent to R_2 and r , $L_{\bar{2}}$ adjacent to $R_2, R_{\bar{2}}, R'_2$ and $\{r_i \neq r\}$. Similarly L'_2 is adjacent to R'_2 and \tilde{r} , $L'_{\bar{2}}$ adjacent to $R'_2, R_{\bar{2}}, R'_2$ and $\{r_i \neq \tilde{r}\}$.

By (ii) there are (unique) vertices $\ell, \tilde{\ell}$ with $t(\ell) = t(r), t(\tilde{\ell}) = t(\tilde{r})$. The adjacencies for $R_2, R'_2, R_{\bar{2}}, R'_{\bar{2}}$ are defined as above: $N(R_2) := \{L_2, \tilde{\ell}\}$, $N(R_{\bar{2}}) := \{L_2, L_{\bar{2}}, L'_{\bar{2}}\} \cup \{\ell_i \neq \tilde{\ell}\}$, $N(R'_2) := \{L'_{\bar{2}}, \ell\}$, $N(R'_{\bar{2}}) := \{L'_2, L_{\bar{2}}, L'_{\bar{2}}\} \cup \{\ell_i \neq \ell\}$.

The adjacencies of the new vertices are well-defined, and the construction is symmetric (as far as degrees and vertex-types go) with respect to L and R , so (ii) holds. The degrees of every ℓ_i increase by 2 (so they now range between at least 3 and at most $2k + 1$). If $t(\ell_i)$ differed from $t(\ell_j)$ in the number of entries equal to d , in B' they will differ in the number of entries equal to $d + 2$. By construction, $t(L_2), t(L'_2), t(L_{\bar{2}})$ and $t(L'_{\bar{2}})$ are distinct from each other (and from the $t(\ell_i)$'s, because of their degrees). Thus (i) holds.

In the bipartite complement, $N(L_2) = \{R_2, R'_2, R'_{\bar{2}}\} \cup \{r_i \neq r\}$, $N(L_{\bar{2}}) = \{R'_2, r\}$, $N(L'_2) = \{R_2, R'_2, R_{\bar{2}}\} \cup \{r_i \neq \tilde{r}\}$ and $N(L'_{\bar{2}}) = \{R_2, \tilde{r}\}$. The neighbourhoods of $R_2, R'_2, R_{\bar{2}}$ and $R'_{\bar{2}}$ are changed similarly. Recall also that $\tilde{t}(\tilde{\ell}) = t(\ell)$ and $\tilde{t}(\tilde{r}) = t(r)$. Thus \tilde{B}' is obtained from \tilde{B} in the same way as we obtained B' from B (with the roles of L_2 and $L'_{\bar{2}}$ interchanged, and similarly for L'_2 and $L_{\bar{2}}$, ℓ and $\tilde{\ell}$, and so on). Since B satisfied (iii), B' does too.

3 Construction on $4k + 1$ vertices

Take a dually vertex-oblique graph G on $4k$ vertices, and introduce a new vertex u_0 that is adjacent to the $2k$ vertices with degree $d \geq 2k$. We claim that the resulting graph G' of order $n' = 4k + 1$ is again dually vertex-oblique. Note that if G was a split graph, then the high-degree vertices must have formed a clique, and thus G' will also be split.

A vertex of degree $d := d_G(v)$ in G has degree $d' := d_{G'}(v)$ in G' . So $d' = d$ if $d < 2k$, and $d' = d + 1$ if $d \geq 2k$. If v and w had complementary degrees in G , that is, $d_G(v) + d_G(w) = n - 1 = 4k - 1$, then $d_{G'}(v) + d_{G'}(w) = n' - 1 = 4k$, so v and w still have complementary degrees in G' ; this means that if $\bar{d} = f$ in G , then $\bar{d}' = f'$ in G' . Also, $d_{G'}(u_0) + d_{G'}(u_0) = 2k + 2k = n' - 1$.

In G' , u_0 will be the unique vertex of degree $2k$; and if v, w , were adjacent in G to different numbers of vertices of degree d , then in G' they are adjacent to different numbers of vertices of degree d' ; so G' is vertex-oblique.

Since G is dually vertex-oblique, for every vertex v there is a unique vertex \bar{v} with $\bar{t}(\bar{v}) = t(v)$. If $t(v) = (x_1, \dots, x_r)$, then \bar{v} must have non-neighbours in G of degrees $\bar{x}_1, \dots, \bar{x}_r$. If $r < 2k$, then u_0 is adjacent to \bar{v} but not to v , so in G' v has vertex-type $t'(v) = (x'_1, \dots, x'_r)$, and \bar{v} has non-neighbours of degrees $\bar{x}'_1, \dots, \bar{x}'_r$; thus $\bar{t}'(\bar{v}) = t'(v)$. If $r > 2k$ then u_0 is adjacent to v in G' , and to \bar{v} in \bar{G}' ; thus $t'(v) = (x'_1, \dots, 2k, \dots, x'_r) = \bar{t}'(\bar{v})$. Finally $t'(u_0) = \bar{t}'(u_0)$, so G' is dually vertex-oblique.

4 Vertex-type sequences: uniqueness and non-uniqueness

The *degree sequence* of a graph on n vertices is the sequence $d_1 \geq \dots \geq d_n$ of its degrees (see footnote 1, p. 2). The *vertex-type sequence* is the sequence $t_1 \succeq \dots \succeq t_n$ of vertex-types, where $t_i \succ t_j$ if t_i is longer than t_j , or if t_i and t_j have the same length and t_i is lexicographically larger than t_j . G_d is the subgraph of G induced by vertices of degree d , and (for $p \neq q$) $G_{p,q}$ is the bipartite subgraph induced by edges joining a vertex of degree p to a vertex of degree q .

Some graphs, such as complete graphs, edgeless graphs and matchings, have unique degree sequence (that is, no other graph has the same degree sequence) and, thus, unique vertex-type sequence. If G is dually vertex-oblique, then by definition its complement shares the same vertex-type sequence, and

is not isomorphic to G because self-complementary graphs have non-trivial automorphisms. But could this complementary pair be the unique graphs with that vertex-type sequence? We show here that the answer is always ‘No’, but that there are infinitely many super vertex-oblique graphs with unique vertex-type sequence.

The key to the proofs is a restricted switching operation. A *switch* is the replacement of edges v_0w_0, v_1w_1 , with new edges v_0w_1, v_1w_0 (that is, v_0w_1 and v_1w_0 did not appear in the original graph); this does not change the degree of any vertex, but may change the vertex-types. A (d, d') -*switch* (or just ‘restricted switch’, when d and d' are not specified) is a switch where v_0 and v_1 both have degree d , and w_0 and w_1 both have degree d' (possibly equal to d); such a switch does not change the type of any vertex. In a bipartite graph, a switch *respects the bipartition* if v_0, v_1 , are in the same part, and (thus) w_0, w_1 , are in the opposite part.

3. Theorem. *For any dually vertex-oblique graph G , there is a graph $H \notin \{G, \overline{G}\}$ with the same vertex-type sequence as G .*

Proof: We will establish:

CLAIM. For any degree $d \neq (n-1)/2$, G has distinct vertices v_0, v_1 , of degree d , and w_0, w_1 , of degree \overline{d} , such that $v_0w_0, v_1w_1 \in E(G)$, $v_0w_1, v_1w_0 \notin E(G)$.

The result follows from the claim since we can then perform a (d, \overline{d}) -switch which gives us another graph H without changing the type of any vertex. Since G has trivial automorphism group, $H \not\cong G$, and we will show that $H \not\cong \overline{G}$.

Let x be any vertex not in $\{v_0, v_1, w_0, w_1\}$. Let \overline{x} be the unique vertex that has the same vertex-type in \overline{G} as x has in G . If $\overline{x} \neq x$ (possibly $\overline{x} \in \{v_0, v_1, w_0, w_1\}$), note that x and \overline{x} are adjacent in H iff they are adjacent in G iff they are *not* adjacent in \overline{G} . If $x = \overline{x}$, take another vertex $y \notin \{x, v_0, v_1, w_0, w_1\}$; note that x and y exist by the remark after Lemma 2. If $y \neq \overline{y}$ we are done, otherwise note that x and y are adjacent in H iff they are adjacent in G iff $\overline{x} = x$ and $\overline{y} = y$ are not adjacent in \overline{G} .

We now turn to proving the Claim, which is equivalent to saying that the bipartite graph $G_{d, \overline{d}}$ has an induced $2K_2$. If there is any vertex

z of degree d that is adjacent to no (or all) vertices of degree \bar{d} , then in \bar{G} z would be a vertex of degree \bar{d} adjacent to all (or no) vertices of degree d , contradicting the fact that G and \bar{G} have the same vertex-type sequence. So in $G_{d,\bar{d}}$ every vertex has at least one neighbour and one non-neighbour from the opposite part.

Let $G_{d,\bar{d}}$ have bipartition $D \cup \bar{D}$. In what follows, x_i will be a vertex in D , $N_i \subseteq \bar{D}$ the set of its neighbours, and $N'_i := \bar{D} \setminus N_i$; $N_i \neq \emptyset \neq N'_i$ by the previous argument. Take an arbitrary vertex $x_0 \in D$. Pick a vertex $y_1 \in N'_0$, and let y_1 be adjacent to some vertex x_1 ; clearly $x_1 \neq x_0$. If there is a vertex $\tilde{y}_0 \in N_0$ such that $\{x_0, x_1, \tilde{y}_0, y_1\}$ induce a $2K_2$, we are done; otherwise, x_1 is adjacent to all of N_0 , as well as y_1 , so $N_1 \supsetneq N_0$, and $N'_1 \subsetneq N'_0$. Pick a vertex $y_2 \in N_1$, and let y_2 be adjacent to x_2 ; as before, $x_2 \neq x_1$, and either there is $\tilde{y}_1 \in N_1$ such that $\{x_1, x_2, \tilde{y}_1, y_2\}$ induce a $2K_2$, or $N'_2 \subsetneq N'_1$. Repeating this procedure we must eventually find an induced $2K_2$, since N'_i can never be empty. \square

A graph G can be transformed by switches into any other graph H with the same degree sequence. If H even has the same vertex-type sequence as G , then we will show how to achieve this using only restricted switches.

Suppose the vertices of a graph G are labeled v_1, \dots, v_n , with $\Delta = d(v_1) \geq \dots \geq d(v_n)$. By switching, we can transform G into a canonical labeled graph G_0 that is determined completely by the degree sequence (the first step in this recursive process is to use switches to make v_1 adjacent to $v_2, \dots, v_{\Delta+1}$); any other labeled graph H with the same vertex-set and the same degree sequence (i.e. $d_G(v_i) = d_H(v_i)$ for all i) can also be transformed into G_0 . These ideas, and analogous ones for bipartite graphs, give us:

Theorem [3, 5]. *If G, H , are two labeled graphs with the same degree sequence, then G can be obtained from H by a sequence of switches. Moreover, if G and H are bipartite, the switches respect the bipartition.* \square

We use this to prove the next result, that has probably also appeared in [11]:

4. Theorem. *If G, H , are two labeled graphs with the same vertex-type sequence, then G can be obtained from H by a sequence of restricted switches.*

Proof: The vertex-type sequence clearly determines the degree sequence. Moreover, for every degree d , the subgraphs G_d and H_d have the same vertex-set and the same degree sequence, since this is also determined by the vertex-types; we can therefore transform G_d into H_d by a sequence of switches; note that in G these are just (d, d) -switches. Similarly, for every $p \neq q$ in the degree sequence, the bipartite graphs $G_{p,q}$ and $H_{p,q}$ have the same vertex-set, the same bipartition, and the same degrees, so we can transform $G_{p,q}$ into $H_{p,q}$ by switches; moreover, we can use switches that respect the bipartition, and these will be valid (p, q) -switches in G even though $G_{p,q}$ is not a vertex-induced subgraph of G . \square

5. Corollary. *If no G_d and no $G_{p,q}$ contains an induced $2K_2$, then G has unique vertex-type sequence. In particular, if every degree appears at most once in G , except for some degree that appears at most three times, then G has unique vertex-type sequence.* \square

The converse of the corollary is not true (the matchings are a counterexample), because a restricted switch may give us a graph G' isomorphic to G . But it can be used to show, for example, that the super vertex-oblique graphs $G_1^6, G_1^7, G_2^8, G_2^9$, in [10] have unique vertex-type sequence. In particular, every degree appears exactly once in G_2^8 , except for five vertices of the same degree that induce a graph with only one edge; it can be checked that applying Construction 1 of [10] with $k = 1$ preserves these properties, and it is shown in that paper that the result is again connected and super vertex-oblique. We thus have:

6. Corollary. *There are infinitely many connected super vertex-oblique graphs with a unique vertex-type sequence.* \square

5 Recognising degree and vertex-type sequences

A graph G *realises* its degree sequence, and its vertex-type sequence. Erdős and Gallai² [4] showed that a sequence $d_1 \geq \dots \geq d_n$ is realised by some

²Several authors have given different characterisations of degree sequences of graphs.

graph if and only if, for $r = 1, \dots, n-1$, we have

$$\sum_{i=1}^r d_i \leq r(r-1) + \sum_{j=r+1}^n \min(r, d_j).$$

If G is a graph with the same degree sequence as \overline{G} , and r_d is the number of vertices of degree d , then:

(*) $d_i + d_{n-i+1} = n-1$, for $i = 1, \dots, n$;

(**) r_d is even for all d , except for $r_{(n-1)/2} \equiv 1 \pmod{4}$.

Clapham and Kleitman [2] showed by construction that every sequence that satisfies (*), (**) and the Erdős-Gallai conditions, is realised by a self-complementary graph. However (Lemma 2.F), not all such sequences are realised by a dually vertex-oblique graph. It would be interesting to characterise the degree sequences of dually vertex-oblique graphs. One might also ask similar questions about vertex-type sequences:

Problem. A. When is a sequence (of sequences of positive integers) the vertex-type sequence of some graph?

B. Characterise the degree sequences and vertex-type sequences of:

- vertex-oblique graphs,
- super vertex-oblique graphs, and
- dually vertex-oblique graphs.

The Erdős-Gallai results on degree-sequences, together with the Gale-Ryser conditions explained below, lead to an efficient algorithm to solve the vertex-type sequence problems; if the sequence is realised by some graph, the algorithm can also be made to construct an example. However, we would like a more succinct characterisation similar to that of Erdős-Gallai, Clapham-Kleitman or Gale-Ryser, especially as this might shed light on the degree sequence problems.

Gale [6] and Ryser [8] showed that sequences $p_1 \geq \dots \geq p_m$ and $q_1 \geq \dots \geq q_n$ are the degrees of a bipartite graph B (with the p_i 's being degrees

on one side, and the q_j 's the degrees on the other side) if and only if, for $r = 1, \dots, n - 1$:

$$\sum_{i=1}^m \min\{r, p_i\} \geq \sum_{j=1}^r q_j.$$

Given the vertex-type sequence of a graph G , we can recover the degree sequence, and compute the vertex-types of \overline{G} (as noted at the beginning of the introduction); it is then straightforward to check whether G is (super or dually) vertex-oblique. So we turn our attention to Problem A.

If we want to check whether a given sequence is actually the vertex-type sequence of some graph G , we recover the degree-sequences of the G_d 's and $G_{p,q}$'s (for all d, p, q , in the degree-sequence of G), and check the Erdős-Gallai and Gale-Ryser conditions, respectively. If the conditions are not all satisfied, we have a contradiction; otherwise, we can construct G_d 's and $G_{p,q}$'s that together give us a graph with the given vertex-type sequence.

6 Other open problems

In a self-complementary graph of order $4k + 1$, one can always remove an appropriate vertex to get a self-complementary graph of order $4k$. It is not clear whether an analogous claim is true for dually vertex-oblique graphs.

Problem. *Is there a dually vertex oblique graph on $4k + 1$ vertices, such that removing any vertex of degree $2k$ leaves a subgraph H such that (a) H does not have the same vertex-types as its complement, or (b) H is not vertex-oblique, or both (a) and (b)?*

For any fixed k , Schreyer et al. [10] constructed super vertex-oblique graphs that were k -connected, with k -connected complements. Our examples of dually vertex-oblique graphs have vertices of degree 2, and thus connectivity at most 2.

Problem. *Are there (complementary pairs of) dually vertex-oblique graphs of arbitrarily high connectivity?*

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